Computational Methods Lecture 1: Computing Equilibrium in a Toy Economy

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Motivation

- When do pen and paper fail in economics?
- Let's start from a simple toy model in which agents interact economically
- As soon as we introduce the a sliver of realism, we need to resort to numerics

A simple toy environment: Primitives

Agents: Alice and Bob

Goods: Apples and Potatoes

Endowments: Alice and Bob each own a given amount of apples and potatoes

- $\bar{x}_{\text{Alice}}$ is the number of apples initially owned by Alice
- $\bar{y}_{\text{Alice}}$ is the number of potatoes initially owned by Alice

A simple toy environment: Preferences

Preferences: Alice and Bob have preferences over apples and potatoes

- We represent those preferences via a **utility functions**

$$u_{\text{Alice}}(x, y) = \alpha_{\text{Alice}} \ln x + (1 - \alpha_{\text{Alice}}) \ln y, \quad \alpha_{\text{Alice}} \in (0, 1).$$

$$u_{\text{Bob}}(x, y) = \alpha_{\text{Bob}} \ln x + (1 - \alpha_{\text{Bob}}) \ln y, \quad \alpha_{\text{Bob}} \in (0, 1).$$

- $u_{\text{Alice}}(x_{\text{Alice}}, y_{\text{Alice}})$ **ordinaly** represents Alice's happiness consuming $(x_{\text{Alice}}, y_{\text{Alice}})$
  - x_{Alice} is the number of apples consumed by Alice
  - y_{Alice} is the number of potatoes consumed by Alice
- This particular functional form is called a **Cobb-Douglas** utility function

Properties of our utility function: Marginal utility

Remember, utility is prescribed by $u(x, y) = \alpha \ln x + (1 - \alpha) \ln y$

We can compute **marginal utility** as

$$\mathrm{MU}_x(x, y) = \frac{\partial u(x, y)}{\partial x} = \frac{\alpha}{x} \quad \text{and} \quad \mathrm{MU}_y(x, y) = \frac{\partial u(x, y)}{\partial y} = \frac{1 - \alpha}{y}$$

- MU_x answers: How much happier does an additional apple make me?
- MU_x is strictly positive: I always value an additional apple
- MU_x is decreasing in x: But less so, when I have lots of apples to begin with

Properties of our utility function: Marginal rate of substitution

Remember, marginal utility is $\mathrm{MU}_x(x, y) = \alpha/x$ and $\mathrm{MU}_y(x, y) = (1-\alpha)/y$

We can now define the **marginal rate of substitution**

$$MRS_{y \to x}(x, y) = \frac{\mathrm{MU}_x(x, y)}{\mathrm{MU}_y(x, y)} = \frac{\frac{\partial u(x, y)}{\partial x}}{\frac{\partial u(x, y)}{\partial y}} = \frac{\frac{\alpha}{x}}{\frac{1-\alpha}{y}}$$

- How many potatoes do you need to give me so I'd be willing to give you one apple? (Think through the units of $MRS_{y \to x}$)
- Note: Every statement here is in infinitesimal terms at the margin...

Why would Alice and Bob trade?

- In autarky (absent any trade), Alice and Bob consume their endowment. That is,

$$x_{\text{Alice}} = \bar{x}_{\text{Alice}} \text{ and } y_{\text{Alice}} = \bar{y}_{\text{Alice}}$$

  as well as

$$x_{\text{Bob}} = \bar{x}_{\text{Bob}} \text{ and } y_{\text{Bob}} = \bar{y}_{\text{Bob}}$$

- We can compute Alice's and Bob's marginal rate of substitution in autarky as

$$\text{MRS}^{\text{Alice}}_{y \to x}(\bar{x}_{\text{Alice}}, \bar{y}_{\text{Alice}}) = \frac{\alpha_{\text{Alice}}}{1 - \alpha_{\text{Alice}}} \frac{\bar{y}_{\text{Alice}}}{\bar{x}_{\text{Alice}}}$$

  and

$$\text{MRS}^{\text{Bob}}_{y \to x}(\bar{x}_{\text{Bob}}, \bar{y}_{\text{Bob}}) = \frac{\alpha_{\text{Bob}}}{1 - \alpha_{\text{Bob}}} \frac{\bar{y}_{\text{Bob}}}{\bar{x}_{\text{Bob}}}$$

Why would Alice and Bob trade?

Let's say that the parameters of the economy $(\alpha, \bar{x}, \bar{y})_{\text{Alice, Bob}}$ are such that

$$\text{MRS}^{\text{Alice}}_{y \to x}(\bar{x}_{\text{Alice}}, \bar{y}_{\text{Alice}}) > \text{MRS}^{\text{Bob}}_{y \to x}(\bar{x}_{\text{Bob}}, \bar{y}_{\text{Bob}})$$

- Alice values potatoes in terms of apples higher than Bob
- Bob should give Alice potatoes in exchange for apples $\implies$ both are happier
  (Equilibrium answers: How many potatoes for how many apples?)

Generally,

$$\text{MRS}^{\text{Alice}}_{y \to x}(\bar{x}_{\text{Alice}}, \bar{y}_{\text{Alice}}) \neq \text{MRS}^{\text{Bob}}_{y \to x}(\bar{x}_{\text{Bob}}, \bar{y}_{\text{Bob}})$$

because their endowments, preferences parameters, or both differ in non-offsetting ways

2. Roadmap

How do we recover equilibrium in this environment?

1. Recover each agent's **demand** from their preferences and budget
   - This gives us quantities as a function of price
2. Use **market clearing** to pin down prices
   - At what price do demand quantities align with available quantities?

Deriving the Budget Constraints: Valuing Endowments

- Let's say each apple costs $\tilde{p}_x$ and each potatoes costs $\tilde{p}_y$ monetary units.
- Alice has $\bar{x}_{\text{Alice}}$ apples valued at $\tilde{p}_x \, \bar{x}_{\text{Alice}}$ and $\bar{y}_{\text{Alice}}$ potatoes valued at $\tilde{p}_y \, \bar{y}_{\text{Alice}}$.
- Since both quantities are in the same units, we can add them together.
- Under prevailing prices $(\tilde{p}_x, \tilde{p}_y)$, the value of Alice's endowment is

$$\tilde{p}_x \, \bar{x}_{\text{Alice}} + \tilde{p}_y \, \bar{y}_{\text{Alice}}$$

Deriving the Budget Constraints: Consumption Choices and Expenditures

- Alice can purchase x_{Alice} apples at $\tilde{p}_x \, x_{\text{Alice}}$ and y_{Alice} potatoes at $\tilde{p}_y \, y_{\text{Alice}}$
- From Alice's perspective, x_{Alice} and y_{Alice} are choice variables
- A choice $(x_{\text{Alice}}, y_{\text{Alice}})$ leads to expenditures

$$\tilde{p}_x \, x_{\text{Alice}} + \tilde{p}_y \, y_{\text{Alice}}$$

Budget Constraint

- Alice can afford $(x_{\text{Alice}}, y_{\text{Alice}})$ only if

$$\tilde{p}_x \, x_{\text{Alice}} + \tilde{p}_y \, y_{\text{Alice}} \leq \tilde{p}_x \, \bar{x}_{\text{Alice}} + \tilde{p}_y \, \bar{y}_{\text{Alice}}$$

- Since there is no point in consuming less than Alice can afford

$$\tilde{p}_x \, x_{\text{Alice}} + \tilde{p}_y \, y_{\text{Alice}} = \tilde{p}_x \, \bar{x}_{\text{Alice}} + \tilde{p}_y \, \bar{y}_{\text{Alice}}$$

- This is Alice's **budget constraint**

Walras' Law

- Alice can afford every combination $(x_{\text{Alice}}, y_{\text{Alice}})$ that satisfies

$$\tilde{p}_x \, x_{\text{Alice}} + \tilde{p}_y \, y_{\text{Alice}} = \tilde{p}_x \, \bar{x}_{\text{Alice}} + \tilde{p}_y \, \bar{y}_{\text{Alice}}$$

- Take any number $\psi > 0$, and multiply it through this budget constraint

$$\psi \, \tilde{p}_x \, x_{\text{Alice}} + \psi \, \tilde{p}_y \, y_{\text{Alice}} = \psi \, \tilde{p}_x \, \bar{x}_{\text{Alice}} + \psi \, \tilde{p}_y \, \bar{y}_{\text{Alice}}$$

- Note that this has no impact on the set of affordable $(x_{\text{Alice}}, y_{\text{Alice}})$
- If the value of Alice's endowment doubles, she can pay double for consumption
- Equilibrium pins down $(\tilde{p}_x, \tilde{p}_y)$ only up to a scale
- As a result, we need a normalization

Normalization: Choosing a Numeraire

- We can say that x (apples) is our **numeraire** and set $\psi = 1/\tilde{p}_x$. So,

$$p_x = \psi\,\tilde{p}_x = 1 \quad \text{and} \quad p_y = \psi\,\tilde{p}_y$$

- This is purely a convention, but we do need to normalize something
- The budget constraint reads

$$x_{\text{Alice}} + p_y\,y_{\text{Alice}} = \bar{x}_{\text{Alice}} + p_y\,\bar{y}_{\text{Alice}}$$

- Now, p_y is the price of potatoes in units of apples
- Tautologically, 1 is the price of apples in units of apples

Constraint Optimization

- Both Alice and Bob want to maximize utility
- Marginal utility is strictly positive for both apples and potatoes $\implies$ consume ∞
- But with finite endowments, they cannot afford infinite consumption
- Their problem is to maximize utility subject to their budget constraint
- Let's see how we do that formally

Maths Digression: Constraint Optimization via Lagrangian

Somewhat loosely speaking, the problem

$$\max_{x,y} \left\{ f(x,y) \text{ subject to } g(x,y) = 0 \right\}$$

is equivalent to the **Lagrangian** problem

$$\max_{x,y,\lambda} \left\{ \mathcal{L}(x,y,\lambda) = f(x,y) - \lambda\, g(x,y) \right\}$$

where we call λ the Lagrange multiplier.

If recover the optimal (x^*, y^*, λ^*) for the second problem

- $f(x^*, y^*)$ is the optimum value of f among all (x,y) such that $g(x,y) = 0$

Framing Alice's Problem as a Lagrangian

- In Alice's problem we have the objective function

$$f(x_{\text{Alice}}, y_{\text{Alice}}) = \alpha_{\text{Alice}} \ln x_{\text{Alice}} + (1 - \alpha_{\text{Alice}}) \ln y_{\text{Alice}}$$

- We can also rewrite Alice's budget constraint to fit the form $g(x, y) = 0$

$$g(x_{\text{Alice}}, y_{\text{Alice}}) = x_{\text{Alice}} + p_y\, y_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y\, \bar{y}_{\text{Alice}}$$

- So the Lagrangian reads

$$\mathcal{L}(x_{\text{Alice}}, y_{\text{Alice}}, \lambda) = \alpha_{\text{Alice}} \ln x_{\text{Alice}} + (1 - \alpha_{\text{Alice}}) \ln y_{\text{Alice}} - \lambda \left(x_{\text{Alice}} + p_y\, y_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y\, \bar{y}_{\text{Alice}} \right)$$

Cobb-Douglas Demand: Program

- Our objective is now a straightforward function in three variables (x, y, λ)

$$\mathcal{L}(x_{\text{Alice}}, y_{\text{Alice}}, \lambda) = \alpha_{\text{Alice}} \ln x_{\text{Alice}} + (1 - \alpha_{\text{Alice}}) \ln y_{\text{Alice}} - \lambda \left(x_{\text{Alice}} + p_y \, y_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y \, \bar{y}_{\text{Alice}} \right)$$

- To maximize, our first step is to take partial derivatives

$$\frac{\partial \mathcal{L}(x_{\text{Alice}}, y_{\text{Alice}}, \lambda)}{\partial x_{\text{Alice}}} = \frac{\alpha_{\text{Alice}}}{x_{\text{Alice}}} - \lambda$$

$$\frac{\partial \mathcal{L}(x_{\text{Alice}}, y_{\text{Alice}}, \lambda)}{\partial y_{\text{Alice}}} = \frac{1 - \alpha_{\text{Alice}}}{y_{\text{Alice}}} - \lambda \, p_y$$

$$\frac{\partial \mathcal{L}(x_{\text{Alice}}, y_{\text{Alice}}, \lambda)}{\partial \lambda} = x_{\text{Alice}} + p_y \, y_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y \, \bar{y}_{\text{Alice}}$$

Cobb-Douglas Demand: First-Order Conditions I

- We then translate these partial derivatives into a system of first-order conditions

$$\begin{aligned}\frac{\alpha_{\text{Alice}}}{x^*_{\text{Alice}}} - \lambda^* &= 0\\ \frac{1-\alpha_{\text{Alice}}}{y^*_{\text{Alice}}} - \lambda^*\, p_y &= 0\\ x^*_{\text{Alice}} + p_y\, y^*_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y\, \bar{y}_{\text{Alice}} &= 0\end{aligned}$$

- Note that this is a system of three equations in three unknown

Cobb-Douglas Demand: First-Order Conditions II

- Rearranging terms a bit

$$\frac{\alpha_{\text{Alice}}}{x^*_{\text{Alice}}} = \lambda^*$$

$$\frac{1 - \alpha_{\text{Alice}}}{y^*_{\text{Alice}}} = \lambda^* \, p_y$$

$$x^*_{\text{Alice}} + p_y \, y^*_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y \, \bar{y}_{\text{Alice}} = 0$$

- This system of equations contains:
  - choices $(x^*_{\text{Alice}}, y^*_{\text{Alice}})$ and (λ^*)
  - exogenous parameters $(\alpha_{\text{Alice}}, \bar{x}_{\text{Alice}}, \bar{y}_{\text{Alice}})$
  - endogenous prices (p_y)
- Our goal is to express choices as a function of prices, while parameters are fixed

Cobb-Douglas Demand: Derivation I

- We first divide the first first-order condition by the second and solve for x^*_{Alice}

$$\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}} \frac{x^*_{\text{Alice}}}{y^*_{\text{Alice}}} = \frac{\lambda^*}{\lambda^*} p_y \qquad \Longleftrightarrow \qquad x^*_{\text{Alice}} = \frac{\alpha_{\text{Alice}}}{1-\alpha_{\text{Alice}}} y^*_{\text{Alice}} p_y$$

- Then, we take the budget constraint (technically our third first-order condition)

$$\color{red}{x^*_{\text{Alice}}} + p_y\, y^*_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y\, \bar{y}_{\text{Alice}} = 0$$

- And substitute in our newly found expression for x^*_{Alice}

$$\color{red}{\frac{\alpha_{\text{Alice}}}{1-\alpha_{\text{Alice}}} y^*_{\text{Alice}} p_y} + p_y\, y^*_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y\, \bar{y}_{\text{Alice}} = 0$$

Cobb-Douglas Demand: Derivation II

- We are left with one equation in one unknown (y^*_{Alice})

$$\frac{\alpha_{\text{Alice}}}{1-\alpha_{\text{Alice}}}\, y^*_{\text{Alice}}\, p_y + p_y\, y^*_{\text{Alice}} - \bar{x}_{\text{Alice}} - p_y\, \bar{y}_{\text{Alice}} = 0$$

- So, let's go through a bit of algebra to solve for y^*_{Alice}

$$\left(\frac{\alpha_{\text{Alice}}}{1-\alpha_{\text{Alice}}} + 1\right) y^*_{\text{Alice}}\, p_y = \bar{x}_{\text{Alice}} + p_y\, \bar{y}_{\text{Alice}}$$

$$\frac{1}{1-\alpha_{\text{Alice}}}\, y^*_{\text{Alice}}\, p_y = \bar{x}_{\text{Alice}} + p_y\, \bar{y}_{\text{Alice}}$$

$$y^*_{\text{Alice}} = (1-\alpha_{\text{Alice}})\, \frac{\bar{x}_{\text{Alice}} + p_y\, \bar{y}_{\text{Alice}}}{p_y}$$

Cobb-Douglas Demand: Derivation III

- We have recovered Alice's demand for potatoes

$$y^*_{\text{Alice}} = (1 - \alpha_{\text{Alice}}) \; \frac{\bar{x}_{\text{Alice}} + p_y \, \bar{y}_{\text{Alice}}}{p_y}$$

- Also, remember that we earlier established that

$$x^*_{\text{Alice}} = \frac{\alpha_{\text{Alice}}}{1 - \alpha_{\text{Alice}}} \, y^*_{\text{Alice}} \, p_y$$

- Substituting in y^*_{Alice}, we find Alice's demand for apples as

$$x^*_{\text{Alice}} = \frac{\alpha_{\text{Alice}}}{1 - \alpha_{\text{Alice}}} \, (1 - \alpha_{\text{Alice}}) \; \frac{\bar{x}_{\text{Alice}} + p_y \, \bar{y}_{\text{Alice}}}{p_y} \; p_y = \alpha_{\text{Alice}} \; \frac{\bar{x}_{\text{Alice}} + p_y \, \bar{y}_{\text{Alice}}}{1}$$

Cobb-Douglas Demand for Apples and Potatoes

- We can go through the exact same steps for Bob to obtain

$$x^*_{\text{Bob}} = \alpha_{\text{Bob}} \; \frac{\bar{x}_{\text{Bob}} + p_y \, \bar{y}_{\text{Bob}}}{1}$$

$$y^*_{\text{Bob}} = (1 - \alpha_{\text{Bob}}) \; \frac{\bar{x}_{\text{Bob}} + p_y \, \bar{y}_{\text{Bob}}}{p_y}$$

- Quick sanity check: Do those demand functions agree with out intuition

$$\text{What I optimally consume} = \text{How much I like it} \times \frac{\text{How much I spend}}{\text{What each unit costs}}$$

Endowments versus Demand

- Total amount of apples available

$$\bar{x} = \bar{x}_{\text{Alice}} + \bar{x}_{\text{Bob}}$$

- Total amount of potatoes available

$$\bar{y} = \bar{y}_{\text{Alice}} + \bar{y}_{\text{Bob}}$$

- Aggregate demand for potatoes at (relative) price p_y

$$y^*_{\text{Alice}} + y^*_{\text{Bob}} = (1 - \alpha_{\text{Alice}}) \ \frac{\bar{x}_{\text{Alice}} + p_y \ \bar{y}_{\text{Alice}}}{p_y} + (1 - \alpha_{\text{Bob}}) \ \frac{\bar{x}_{\text{Bob}} + p_y \ \bar{y}_{\text{Bob}}}{p_y}$$

Aggregate Demand for Potatoes: Some Algebra

- First we can multiply out the parentheses

$$y^*_{\text{Alice}} + y^*_{\text{Bob}} = \frac{(\bar{x}_{\text{Alice}} + \bar{x}_{\text{Bob}} - \alpha_{\text{Alice}}\,\bar{x}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{x}_{\text{Bob}}) + p_y\,(\bar{y}_{\text{Alice}} + \bar{y}_{\text{Bob}} - \alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}})}{p_y}$$

- Then, using $\bar{x} = \bar{x}_{\text{Alice}} + \bar{x}_{\text{Bob}}$ and $\bar{y} = \bar{y}_{\text{Alice}} + \bar{y}_{\text{Bob}}$

$$y^*_{\text{Alice}} + y^*_{\text{Bob}} = \frac{(\bar{x} - \alpha_{\text{Alice}}\,\bar{x}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{x}_{\text{Bob}}) + p_y\,(\bar{y} - \alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}})}{p_y}$$

Market Clearing

- Our toy economy is in equilibrium when markets clear

$$y^*_{\text{Alice}} + y^*_{\text{Bob}} = \bar{y}$$

- We substitute in our aggregate demand expression

$$\frac{(\bar{x} - \alpha_{\text{Alice}}\,\bar{x}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{x}_{\text{Bob}}) + p_y\,(\bar{y} - \alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}})}{p_y} = \bar{y}$$

- This equation only features parameters and the relative price of potatoes p_y

- Our goal was to find the **price** at which markets clear given that agents optimize
  Put differently, solve this equation for p_y

Market Clearing: Some Algebra

- To solve for p_y, let's go through some algebra

$$(\bar{x} - \alpha_{\text{Alice}}\,\bar{x}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{x}_{\text{Bob}}) + p_y\,(\bar{y} - \alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}}) = p_y\,\bar{y}$$

$$(\bar{x} - \alpha_{\text{Alice}}\,\bar{x}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{x}_{\text{Bob}}) = p_y\left(\bar{y} - (\bar{y} - \alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}})\right)$$

$$(\bar{x} - \alpha_{\text{Alice}}\,\bar{x}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{x}_{\text{Bob}}) = p_y\,(\alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} + \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}})$$

$$p_y = \frac{\bar{x} - \alpha_{\text{Alice}}\,\bar{x}_{\text{Alice}} - \alpha_{\text{Bob}}\,\bar{x}_{\text{Bob}}}{\alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} + \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}}}$$

Equilibrium

- The **equilibrium price** of potatoes in terms of apples is

$$p_y = \frac{(1-\alpha_{\text{Alice}})\,\bar{x}_{\text{Alice}} + (1-\alpha_{\text{Bob}})\,\bar{x}_{\text{Bob}}}{\alpha_{\text{Alice}}\,\bar{y}_{\text{Alice}} + \alpha_{\text{Bob}}\,\bar{y}_{\text{Bob}}}$$

- Intuitively, the price of potatoes is higher if
  - Potatoes are better liked: $(\alpha_{\text{Alice}}, \alpha_{\text{Bob}})$ are low
  - Potatoes are less plentiful: $(\bar{y}_{\text{Alice}}, \bar{y}_{\text{Bob}})$ is low relative to $(\bar{x}_{\text{Alice}}, \bar{x}_{\text{Bob}})$

Is that success?

- We did all of this with pen and paper
- But we made ridiculously restrictive assumptions
  - Two agents
  - Two goods
  - No savings
  - No technology
  - No labor
  - Unit elasticity of substitution
  - ...
- Let's start simple and relax the latter

The Elasticity of Substitution

- We define the **elasticity of substitution** as

$$\sigma \equiv \frac{d \log(y/x)}{d \log(p_x/p_y)}$$

- This object answers the following question
  - The relative price of apples increases by 1%
  - By how many percent does my relative demand for potatoes increase
- This could be very different for different pairs of goods
  - Apples and potatoes
  - Clementines and tangerines
  - Right and left shoes

Cobb-Douglas: Elasticity of Substitution I

- We defined the **elasticity of substitution** as

$$\sigma \equiv \frac{d \log(y/x)}{d \log(p_x/p_y)}$$

- For Alice with Cobb-Douglas preferences, her demands are

$$y^*_{\text{Alice}} = (1 - \alpha_{\text{Alice}}) \ \frac{p_x \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}}{p_y}$$

$$x^*_{\text{Alice}} = \alpha_{\text{Alice}} \ \frac{p_x \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}}{p_x}$$

  where $p_x = 1$ (apples are the numeraire)

- Next, we compute the ratio $y^*_{\text{Alice}}/x^*_{\text{Alice}}$ and see how it responds to prices

Cobb-Douglas: Elasticity of Substitution II

- Take the ratio of demands

$$\frac{y^*_{\text{Alice}}}{x^*_{\text{Alice}}} = \frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}} \frac{p_x \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}}{p_x \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}} \frac{p_x}{p_y}.$$

- Cancel $p_x \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}$ in numerator and denominator

$$\frac{y^*_{\text{Alice}}}{x^*_{\text{Alice}}} = \frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}} \frac{p_x}{p_y}.$$

- Take logs of the ratio

$$\log\left(\frac{y^*_{\text{Alice}}}{x^*_{\text{Alice}}}\right) = \log\left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) + \log\left(\frac{p_x}{p_y}\right).$$

Cobb-Douglas: Elasticity of Substitution III

- So, let's take
$$\log\left(\frac{y^*_{\text{Alice}}}{x^*_{\text{Alice}}}\right) = \log\left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) + \log\left(\frac{p_x}{p_y}\right)$$
- And differentiate with respect to $\log(p_x/p_y)$
$$\sigma \equiv \frac{d\log(y^*_{\text{Alice}}/x^*_{\text{Alice}})}{d\log(p_x/p_y)} = 1$$
- **Conclusion:** Cobb–Douglas preferences *force* an elasticity of substitution of 1

Why move beyond Cobb-Douglas?

- In many applications, goods are *not* unit-elastic substitutes:
  - Left and right shoes behave more like **complements**: $\sigma \ll 1$
  - Clementines and tangerines are extremely close **substitutes**: $\sigma \gg 1$
  - Apples and potatoes are plausibly moderately substitutable
- Cobb-Douglas builds in $\sigma = 1$ by construction:
  - It cannot capture "tighter" or "weaker" substitution patterns.
  - This can be quite restrictive when matching data.
- We want a simple but flexible class where the elasticity of substitution σ is free

Constant Elasticity of Substitution (CES) Preferences

- We can define the CES utility function for Alice and Bob as

$$u_{\text{Alice}}(x,y) = \left(\alpha_{\text{Alice}}^{\frac{1}{\sigma}} x^{\frac{\sigma-1}{\sigma}} + (1-\alpha_{\text{Alice}})^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$
$$u_{\text{Bob}}(x,y) = \left(\alpha_{\text{Bob}}^{\frac{1}{\sigma}} x^{\frac{\sigma-1}{\sigma}} + (1-\alpha_{\text{Bob}})^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

  where $\sigma > 0$ and $\sigma \neq 1$

- Here σ parametrizes the constant **elasticity of substitution** between x and y

- $(\alpha_{\text{Alice}}, \alpha_{\text{Bob}}) \in (0,1)^2$ still encodes the relative taste for apples vs. potatoes

Limiting cases of CES

- As $\sigma \to 1$ (use L'Hôpital), we recover Cobb-Douglas:

$$\lim_{\sigma \to 1} \left(\alpha^{\frac{1}{\sigma}} x^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = x^{\alpha} y^{1-\alpha}$$

- As $\sigma \to \infty$, goods become perfect substitutes:

$$\lim_{\sigma \to \infty} \left(\alpha^{\frac{1}{\sigma}} x^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = x + y$$

- As $\sigma \to 0$, goods become perfect complements:

$$\lim_{\sigma \to 0} \left(\alpha^{\frac{1}{\sigma}} x^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \min\{x, y\}$$

- So CES nests Cobb-Douglas and the standard extremes in a single unified class

Alice's Optimization Problem

- Utility is now given as

$$u_{\text{Alice}}(x, y) = \left(\alpha_{\text{Alice}}^{\frac{1}{\sigma}} x^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_{\text{Alice}})^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

- And the budget constraint reads

$$p_x x + p_y y = p_x \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}$$

Alice's Lagrangian

- We can now define the inner CES aggregator

$$G(x, y) \equiv \alpha_{\text{Alice}}^{\frac{1}{\sigma}} x^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_{\text{Alice}})^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}}$$

such that $u_{\text{Alice}}(x, y) = G(x, y)^{\frac{\sigma}{\sigma-1}}$

- The Lagrangian can then be written as

$$\mathcal{L}(x, y, \lambda) = G(x, y)^{\frac{\sigma}{\sigma-1}} - \lambda\Big(p_x x + p_y y - p_x \bar{x}_{\text{Alice}} - p_y \bar{y}_{\text{Alice}}\Big)$$

CES Demand: Marginal Utilities

- Differentiate G with respect to x

$$\frac{\partial G}{\partial x} = \alpha_{\text{Alice}}^{\frac{1}{\sigma}} \frac{\sigma - 1}{\sigma} x^{\frac{\sigma-1}{\sigma} - 1}$$

- Now, since $u = G^{\frac{\sigma}{\sigma-1}}$ the chain rule yields

$$\frac{\partial u}{\partial x} = \frac{\sigma}{\sigma - 1} G^{\frac{\sigma}{\sigma-1} - 1} \frac{\partial G}{\partial x}$$

- Substituting in the partial derivative of G with respect to x

$$\frac{\partial u}{\partial x} = \frac{\sigma}{\sigma - 1} G^{\frac{1}{\sigma-1}} \alpha_{\text{Alice}}^{\frac{1}{\sigma}} \frac{\sigma - 1}{\sigma} x^{\frac{\sigma-1}{\sigma} - 1} = G^{\frac{1}{\sigma-1}} \alpha_{\text{Alice}}^{\frac{1}{\sigma}} x^{-\frac{1}{\sigma}}$$

- Analogously,

$$\frac{\partial u}{\partial y} = G^{\frac{1}{\sigma-1}} (1 - \alpha_{\text{Alice}})^{\frac{1}{\sigma}} y^{-\frac{1}{\sigma}}$$

CES Demand: First-Order Conditions

- From the Lagrangian

$$\mathcal{L}(x, y, \lambda) = u_{\text{Alice}}(x, y) - \lambda\left(p_x x + p_y y - p_x \bar{x}_{\text{Alice}} - p_y \bar{y}_{\text{Alice}}\right)$$

- we obtain our familiar system of three equations in three unknowns

$$\frac{\partial \mathcal{L}}{\partial x} = G^{\frac{1}{\sigma-1}} \alpha_{\text{Alice}}^{\frac{1}{\sigma}} x^{-\frac{1}{\sigma}} - \lambda p_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = G^{\frac{1}{\sigma-1}} (1 - \alpha_{\text{Alice}})^{\frac{1}{\sigma}} y^{-\frac{1}{\sigma}} - \lambda p_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_x x + p_y y - p_x \bar{x}_{\text{Alice}} - p_y \bar{y}_{\text{Alice}} = 0$$

- Next step: eliminate G and λ by using ratios of FOCs.

CES Demand: Solving for Relative Demand y/x

- First, we divide the first two FOCs

$$\frac{G^{\frac{1}{\sigma-1}}\alpha_{\text{Alice}}^{\frac{1}{\sigma}}x^{-\frac{1}{\sigma}}}{G^{\frac{1}{\sigma-1}}(1-\alpha_{\text{Alice}})^{\frac{1}{\sigma}}y^{-\frac{1}{\sigma}}} = \frac{\lambda p_x}{\lambda p_y} = \frac{p_x}{p_y}$$

- We can then simplify the left-hand side

$$\left(\frac{\alpha_{\text{Alice}}}{1-\alpha_{\text{Alice}}}\right)^{\frac{1}{\sigma}}\left(\frac{y}{x}\right)^{\frac{1}{\sigma}} = \frac{p_x}{p_y}$$

- And raise both sides to the power σ

$$\frac{\alpha_{\text{Alice}}}{1-\alpha_{\text{Alice}}}\frac{y}{x} = \left(\frac{p_x}{p_y}\right)^{\sigma} \quad \Rightarrow \quad \frac{y}{x} = \left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right)\left(\frac{p_x}{p_y}\right)^{\sigma}$$

CES Demand: Elasticity of Substitution

- We found the demand ratio

$$\frac{y}{x} = \left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) p_y^{-\sigma} = \left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) \left(\frac{p_x}{p_y}\right)^{\sigma}$$

- Take logs

$$\log\left(\frac{y}{x}\right) = \log\left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) + \sigma \log\left(\frac{p_x}{p_y}\right).$$

- Differentiate with respect to $\log(p_x/p_y)$

$$\frac{d\log(y/x)}{d\log(p_x/p_y)} = \sigma$$

- **Conclusion:** The elasticity of substitution is exactly the parameter σ

CES Demand: using the budget constraint

- Let's choose apples as our numeraire and set $p_x = 1$ such that

$$\frac{y}{x} = \left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) p_y^{-\sigma}$$

gives us

$$y = \left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) p_y^{-\sigma} x$$

- Plugging this into the budget constraint

$$x + p_y y = \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}$$

gives us

$$x + p_y \left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right) p_y^{-\sigma} x = \bar{x}_{\text{Alice}} + p_y \bar{y}_{\text{Alice}}$$

CES Demand: levels of x and y

- Rearranging terms a bit

$$x\left(1+p_y\left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right)\,p_y^{-\sigma}\right)=\bar{x}_{\text{Alice}}+p_y\,\bar{y}_{\text{Alice}}$$

we obtain Alice's demand for apples

$$x^*_{\text{Alice}}(p_y)=\frac{\bar{x}_{\text{Alice}}+p_y\,\bar{y}_{\text{Alice}}}{1+p_y\left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right)\,p_y^{-\sigma}}=\frac{\alpha_{\text{Alice}}\left(\bar{x}_{\text{Alice}}+p_y\,\bar{y}_{\text{Alice}}\right)}{\alpha_{\text{Alice}}+(1-\alpha_{\text{Alice}})\;p_y^{1-\sigma}}$$

- And, in turn, Alice's demand for potatoes

$$y^*_{\text{Alice}}(p_y)=\left(\frac{1-\alpha_{\text{Alice}}}{\alpha_{\text{Alice}}}\right)\,p_y^{-\sigma}\,x^*_{\text{Alice}}(p_y)=\frac{(1-\alpha_{\text{Alice}})\,p_y^{-\sigma}\left(\bar{x}_{\text{Alice}}+p_y\,\bar{y}_{\text{Alice}}\right)}{\alpha_{\text{Alice}}+(1-\alpha_{\text{Alice}})\;p_y^{1-\sigma}}$$

CES Demand: Bob

- Analogously, we obtain Bob's demand for apples

$$x^*_{\text{Bob}}(p_y) = \frac{\alpha_{\text{Bob}} \left(\bar{x}_{\text{Bob}} + p_y \bar{y}_{\text{Bob}}\right)}{\alpha_{\text{Bob}} + (1 - \alpha_{\text{Bob}})\ p_y^{1-\sigma}}$$

- And Bob's demand for potatoes

$$y^*_{\text{Bob}}(p_y) = \frac{(1 - \alpha_{\text{Bob}})\, p_y^{-\sigma} \left(\bar{x}_{\text{Bob}} + p_y \bar{y}_{\text{Bob}}\right)}{\alpha_{\text{Bob}} + (1 - \alpha_{\text{Bob}})\ p_y^{1-\sigma}}$$

- This mirrors Alice's formulas one-for-one

Market Clearing: Endowments and Aggregate Demand

- As before, the total amount of apples and potatoes in the economy are

$$\bar{x} = \bar{x}_{\text{Alice}} + \bar{x}_{\text{Bob}} \quad \text{and} \quad \bar{y} = \bar{y}_{\text{Alice}} + \bar{y}_{\text{Bob}}$$

- We can write down the aggregate demand for potatoes at price p_y as

$$y^*_{\text{Alice}}(p_y) + y^*_{\text{Bob}}(p_y)$$

- Market clearing for potatoes requires

$$y^*_{\text{Alice}}(p_y) + y^*_{\text{Bob}}(p_y) = \bar{y}$$

- This is one non-linear equation in the single unknown p_y

Market Clearing: Role for Numerics I

- Let's write this out explicitly

$$\frac{(1-\alpha_{\text{Alice}})\, p_y^{1-\sigma}}{\alpha_{\text{Alice}}+(1-\alpha_{\text{Alice}})\, p_y^{1-\sigma}}\ \frac{\bar{x}_{\text{Alice}}+p_y\,\bar{y}_{\text{Alice}}}{p_y} + \frac{(1-\alpha_{\text{Bob}})\, p_y^{1-\sigma}}{\alpha_{\text{Bob}}+(1-\alpha_{\text{Bob}})\, p_y^{1-\sigma}}\ \frac{\bar{x}_{\text{Bob}}+p_y\,\bar{y}_{\text{Bob}}}{p_y} = \bar{y}$$

- This mixes p_y and $p_y^{1-\sigma}$ inside rational expressions
- In general, there is **no simple closed-form solution** for p_y
- But we can define the excess demand function

$$F(p_y) = y^*_{\text{Alice}}(p_y) + y^*_{\text{Bob}}(p_y) - \bar{y}$$

- The equilibrium price p_y^* solves $F(p_y^*) = 0$

Market Clearing: Role for Numerics II

- If you want to convince yourself, have a stab at solving

$$F(p_y^*) = 0$$

- Pen and paper are not going to get you far
- But it is still a single equation in a single unknown
- The fact that we cannot find p_y^* with pen and paper, doesn't mean p_y^* does not exist
- This is non-linear root-finding problem for which we need **numerical methods**