ECO 612 Lecture 1:
Incomplete Markets and Aiyagari

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Motivation

Why Incomplete Markets?

- **Idiosyncratic** income shocks
- Missing state-contingent insurance
- Borrowing limits
- Self-insurance via **buffer-stock** assets

What incomplete markets deliver

- Non-degenerate wealth distribution in steady state
- Precautionary saving emerges from risk + constraints
- Micro risk and constraints pin down aggregate capital (and thus r, w)

What Gets “Turned Off” in Complete Markets?

- Full insurance of **idiosyncratic** risk
- Consumption responds only to **aggregate** shocks
- Little role for wealth dispersion as a smoothing device

A practical contrast

- **Complete markets:** a bad idiosyncratic shock is insured $\Rightarrow$ no need to run down assets
- **Incomplete markets:** a bad idiosyncratic shock hits consumption unless you have a buffer $\Rightarrow$ assets matter
- Result: the savings rule (and stationary distribution) becomes central

A Minimal Environment

- Infinite-lived households with CRRA preferences
- Idiosyncratic labor productivity z_t follows a Markov process
- One risk-free asset a with a borrowing limit
- Competitive firms rent capital and hire labor

Household Problem

Preferences and Shocks

- Utility: $\sum_{t=0}^{\infty} \beta^t u(c_t)$
- CRRA: $u(c) = \dfrac{c^{1-\gamma} - 1}{1 - \gamma}$
- Idiosyncratic state $z_t \in \mathcal{Z}$ follows a Markov chain

A common income process

- $\log z_{t+1} = \rho_z \log z_t + \sigma_z \, \varepsilon_{t+1}$ and $\varepsilon_{t+1} \sim \mathcal{N}(0, 1)$
- Discretize z into a finite Markov chain $(\mathcal{Z}, \Pi)$

Budget Constraint and Feasibility

Let prices be constant (r, w). Assets are one-period risk-free.

- Cash-on-hand: $x = wz + (1+r)\,a$
- Choice: next assets a'
- Consumption: $c = x - a'$
- Borrowing limit: $a' \geq \underline{a}$

Key constraint (where incompleteness bites)

- Feasibility: $c = wz + (1+r)\,a - a'$
- Nonnegativity: $c \geq 0$
- Borrowing: $a' \geq \underline{a}$

Sequence Space View

Household Problem in Sequence Space

Fix a realized path $\{z_t\}$ and constant prices (r, w).

- Choose sequences $\{c_t, a_{t+1}\}_{t \geq 0}$
- Subject to the budget constraint for all t
- Subject to the borrowing constraint for all t

Sequence formulation

- The household seeks to maximize

$$\max_{\{c_t, a_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad c_t + a_{t+1} = w z_t + (1+r)\, a_t, \quad a_{t+1} \geq \underline{a}, \quad c_t \geq 0.$$

Euler Equation with a Borrowing Constraint

The borrowing constraint generates **inequality** conditions.

- Let $\mu_t \geq 0$ be the multiplier on $a_{t+1} \geq \underline{a}$
- The Euler equation becomes a complementarity condition

KKT Euler (workhorse)

- Euler inequality:
$$u'(c_t) \;\geq\; \beta\,(1+r)\,\mathbb{E}[u'(c_{t+1}) \mid z_t]\,.$$
- Complementary slackness:
$$\Big(u'(c_t) - \beta\,(1+r)\,\mathbb{E}[u'(c_{t+1}) \mid z_t]\Big)\,\Big(a_{t+1} - \underline{a}\Big) = 0.$$

Consumption Smoothing

- If the constraint never binds, the Euler holds with equality
- If the constraint binds, consumption falls "too much" today
- Therefore c_t is smoother than income, but not flat

Benchmarks

When is $r = \frac{1}{\beta} - 1$?

Consider a one-period *risk-free* asset with gross return $1 + r$.

- Interior optimum (constraint slack) implies the Euler equality
- The risk-free rate is pinned down by the expected SDF

Euler $\Rightarrow$ condition for $1 + r = \beta^{-1}$

- $$u'(c_t) = \beta\,(1+r)\,\mathbb{E}_t[u'(c_{t+1})] \quad \Longrightarrow \quad 1 + r = \frac{1}{\beta\,\mathbb{E}_t[u'(c_{t+1})/u'(c_t)]}.$$
- Hence
$$r = \frac{1}{\beta} - 1 \quad \Longleftrightarrow \quad \mathbb{E}_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] = 1.$$

When is $r = \frac{1}{\beta} - 1$? Sufficient Cases

Clean sufficient conditions

- **Representative agent in deterministic steady state:**

$$c_{t+1} = c_t \rightsquigarrow 1 + r = \frac{1}{\beta}$$

- **Complete markets + no aggregate risk:**

$$\text{idiosyncratic shocks insured} \rightsquigarrow \text{constant } c_t \rightsquigarrow 1 + r = \tfrac{1}{\beta}$$

- **Linear utility:**

$$u(c) = \alpha_0 + \alpha_1\, c \Rightarrow u'(c) = \alpha_1 \rightsquigarrow 1 + r = \tfrac{1}{\beta}$$

Why Aiyagari typically has $r < \frac{1}{\beta} - 1$

- Risk-free Euler for the **marginal unconstrained** saver (so equality holds)

$$u'(c_t) = \beta\,(1+r)\,\mathbb{E}_t[u'(c_{t+1})] \qquad \Rightarrow \qquad 1 + r = \frac{u'(c_t)}{\beta\,\mathbb{E}_t[u'(c_{t+1})]}.$$

- Uninsured idiosyncratic risk + prudence imply $u'(c)$ is convex, so

$$\mathbb{E}_t[u'(c_{t+1})] \; > \; u'\big(\mathbb{E}_t[c_{t+1}]\big)$$

  which reflects that bad states get extra weight.

- Therefore, whenever the pricer has little expected consumption growth

$$\mathbb{E}_t[c_{t+1}] \approx c_t \quad \text{so that} \quad \mathbb{E}_t[u'(c_{t+1})] > u'(c_t)$$

  and the safe return must fall below $\frac{1}{\beta} - 1$.

LQ implies affine $c(a, z)$

- Quadratic utility (interior):

$$u(c) = c - \frac{\gamma}{2}c^2 \qquad \Rightarrow \qquad u'(c) = 1 - \gamma c.$$

- Euler (risk-free, no binding constraints), with $R \equiv 1 + r$:

$$1 - \gamma c_t = \beta\,(1+r)\,\mathbb{E}_t[1 - \gamma c_{t+1}] \qquad \Rightarrow \qquad c_t = \frac{1 - \beta\,(1+r)}{\gamma} + \beta\,(1+r)\,\mathbb{E}_t[c_{t+1}].$$

- Budget and income process:

$$c_t + a_{t+1} = (1+r)\,a_t + w\,z_t \quad \text{and} \quad z_{t+1} = \rho z_t + \varepsilon_{t+1}$$

so

$$\mathbb{E}_t[a_{t+1}] = (1+r)\,a_t + z_t - c_t \quad \text{and} \quad \mathbb{E}_t[z_{t+1}] = \rho z_t.$$

LQ implies affine $c(a, z)$

- Conjecture affine policy: $c_t = \kappa_0 + \kappa_a a_t + \kappa_z z_t$. Then

$$\mathbb{E}_t[c_{t+1}] = \kappa_0 + \kappa_a \, \mathbb{E}_t[a_{t+1}] + \kappa_z \, \mathbb{E}_t[z_{t+1}] = \kappa_0 + \kappa_a \big((1+r)\, a_t + w\,_t - c_t \big) + \kappa_z \rho z_t.$$

- Plug into $c_t = \frac{1-\beta\,(1+r)}{\gamma} + \beta\,(1+r)\; \mathbb{E}_t[c_{t+1}]$ and collect c_t:

$$(1{+}\beta\,(1{+}r)\,\kappa_a)\,c_t = \frac{1-\beta\,(1+r)}{\gamma} {+} \beta\,(1{+}r)\,\kappa_0 {+} \beta\,(1{+}r)^2 \kappa_a\, a_t {+} \beta\,(1{+}r)\,\big(\kappa_a {+} \rho\kappa_z\big)\, w\, z_t.$$

- Divide:

$$c_t = \frac{\frac{1-\beta\,(1+r)}{\gamma} + \beta\,(1+r)\kappa_0}{1+\beta\,(1+r)\kappa_a} + \frac{\beta\,(1+r)^2\kappa_a}{1+\beta\,(1+r)\kappa_a}\, a_t + \frac{\beta\,(1+r)\,(\kappa_a+\rho\kappa_z)}{1+\beta\,(1+r)\kappa_a}\, w\, z_t$$

so $c(a, z)$ is affine.

Bellman Equation

Recursive Formulation

State is (a, z). Choice is a'.

- The Bellman equation restates the same problem recursively
- This is what we solve numerically

Bellman equation (prices taken as given)

- Value function:

$$V(a,z) = \max_{a' \geq \underline{a}} \left\{ u\big(wz + (1+r)\,a - a'\big) + \beta\,\mathbb{E}[V(a',z') \mid z] \right\}.$$

- Consumption is residual: $c = wz + (1+r)\,a - a'$ (and $c \geq 0$ implicitly)

Digression: Cash-on-Hand Formulation

Define cash-on-hand $m \equiv wz + (1+r)\,a$.

Bellman in (m, z)

- Value function:
$$V(m,z) = \max_{a' \geq \underline{a}} \left\{ u(m - a') + \beta\,\mathbb{E}[V(m', z') \mid z] \right\}.$$
- Laws of motion:
$$m' = wz' + (1+r)\,a' \quad \text{and} \quad z' \sim \Pi(\cdot \mid z).$$

Policy Functions

Once V is solved, we get rules.

- Savings rule: $a'(a, z)$
- Consumption rule: $c(a, z) = wz + (1 + r)\, a - a'(a, z)$
- These rules generate the steady-state distribution μ

Or as a function of m if using a cash-in-hand formulation.

MPCs in Incomplete Markets: Definition

Use cash-on-hand $m = wz + (1+r)\,a$ and the policy $a'(m,z)$.

Individual MPC

- Consumption policy: $c(m,z) = m - a'(x,z)$.
- Marginal propensity to consume out of a transitory dollar:

$$\mathrm{MPC}(m,z) \equiv \frac{\partial c(m,z)}{\partial m} = 1 - \frac{\partial a'(m,z)}{\partial m}.$$

- At the borrowing constraint $a'(m,z) = \underline{a}$:

$$\frac{\partial a'}{\partial m} = 0 \;\Rightarrow\; \mathrm{MPC}(m,z) = 1.$$

MPCs in Incomplete Markets: Aggregation and Intuition

Aggregate MPC is an expectation under the stationary distribution.

Aggregate MPC

- Stationary aggregate MPC:

$$\overline{\text{MPC}} \equiv \int \text{MPC}(m, z)\, \mu(dm, dz).$$

- Mechanism:

$$\text{More mass near } a' = \underline{a} \;\Rightarrow\; \overline{\text{MPC}} \uparrow .$$

- Comparative statics: tighter $\underline{a}$ or higher σ_z typically increase $\overline{\text{MPC}}$

Prudence and Precautionary Saving

Prudence $\Rightarrow$ convex marginal utility

- Risk aversion: $u''(c) < 0$ so $u'(c)$ is decreasing in c.
- Prudence: $u'''(c) > 0$ so $u'(c)$ is *convex*:

$$u'''(c) > 0 \quad \Leftrightarrow \quad u'(\cdot) \text{ convex.}$$

- Measure of prudence (curvature of u' scaled by curvature of u):

$$P(c) \equiv -\frac{u'''(c)}{u''(c)} \; > \; 0.$$

- CRRA example $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$u'(c) = c^{-\gamma}, \qquad P(c) = \frac{\gamma + 1}{c}.$$

Convex marginal utility $\Rightarrow$ precautionary saving

- Interior Euler: $u'(c_t) = \beta\,(1+r)\,\mathbb{E}_t[u'(c_{t+1})]$.
- By Jensen's inequality risk increases $\mathbb{E}_t[u'(c_{t+1})]$.
- Holding r fixed, Euler equality then requires $u'(c_t)$ to rise.
- Because u' is decreasing ($u'' < 0$), this means

$$u'(c_t)\uparrow \quad \Rightarrow \quad c_t \downarrow .$$

- With cash-on-hand x_t fixed and $c_t + a_{t+1} = x_t$:

$$c_t \downarrow \quad \Rightarrow \quad a_{t+1} \uparrow .$$

- In GE, higher desired saving bids up the bond and pushes R down.

Stationary Distribution

From Policies to a Distribution

Aiyagari is about a **distribution** over (a, z).

- Given $a'(a, z)$ and $z' \sim \Pi(\cdot \mid z)$
- (a_{t+1}, z_{t+1}) is a Markov process
- We look for an invariant distribution μ

Invariant distribution

- Fixed point: $\mu = \mathcal{T}(\mu)$

Constructing the Transition Kernel

Let $\mathcal{S} = \mathcal{A} \times \mathcal{Z}$. It suffices to specify transitions on rectangles $A \times Z$.

Kernel and push-forward

- Rectangle transition probability:

$$Q\big((a,z), A \times Z\big) = \Pr\{a'(a,z) \in A,\ z' \in Z \mid z\} = \mathbf{1}\{a'(a,z) \in A\}\, \Pi(z, Z).$$

- Distribution update:

$$(\mathcal{T}\mu)(A \times Z) = \int_{\mathcal{A} \times \mathcal{Z}} Q\big((a,z), A \times Z\big)\, \mu(da, dz).$$

- Invariance:

$$\mu(A \times Z) = (\mathcal{T}\mu)(A \times Z) \quad \forall\, A, Z.$$

Discrete Case: Transition Matrix Intuition

If a and z live on grids, you can build a big transition matrix.

- States: (a_i, z_j) for $i = 1, \ldots, N_a$, $j = 1, \ldots, N_z$
- Policy maps to off-grid a': use interpolation weights
- μ is a stationary probability vector: $\mu = \mu P$

What you actually compute

- Iterate forward:

$$\mu^{(k+1)} = \mu^{(k)} P$$

- Stop when $\|\mu^{(k+1)} - \mu^{(k)}\|$ is tiny

Steady-State Objects You Care About

Once you have $\mu(a, z)$, everything aggregates.

- Aggregate assets: $A = \int a \, d\mu(a, z)$
- Consumption: $C = \int c(a, z) \, d\mu(a, z)$
- Wealth inequality: Gini, top shares

Aiyagari General Equilibrium

Production Side

Representative firm with Cobb–Douglas production.

- Production: $Y = K^\alpha L^{1-\alpha}$
- Firm takes (r, w) and chooses K, L
- Competitive factor prices equal marginal products

Factor prices

- Profit-seeking firm behavior is imposed by requiring that

$$r = \alpha \left(\frac{K}{L}\right)^{\alpha-1} - \delta \quad \text{and} \quad w = (1-\alpha)\left(\frac{K}{L}\right)^{\alpha}.$$

Stationary Competitive Equilibrium: The Fixed Point

A stationary equilibrium pins down (r, w), policies, and the distribution.

- Given (r, w), households solve for $a'(a, z)$
- $a'(a, z)$ and Π imply an invariant $\mu(a, z)$
- Aggregates from μ must reproduce (r, w) via firm FOCs

One-dimensional root-finding (common implementation)

- Given a candidate r, compute $w = w(r)$ from firm FOCs (using L normalization)
- Solve households $\Rightarrow \mu_r \Rightarrow$ implied $K(r) = \int a \, d\mu_r$
- Update r until $K(r)$ is consistent with firm demand

Stationary Competitive Equilibrium: Market Clearing

Market clearing and price consistency

- **Capital market:**

$$K = \int_{\mathcal{A}\times\mathcal{Z}} a\,\mu(da, dz)$$

- **Labor market:**

$$L = \int_{\mathcal{A}\times\mathcal{Z}} z\,\mu(da, dz) \quad \text{(often normalized)}$$

- **Price consistency:**

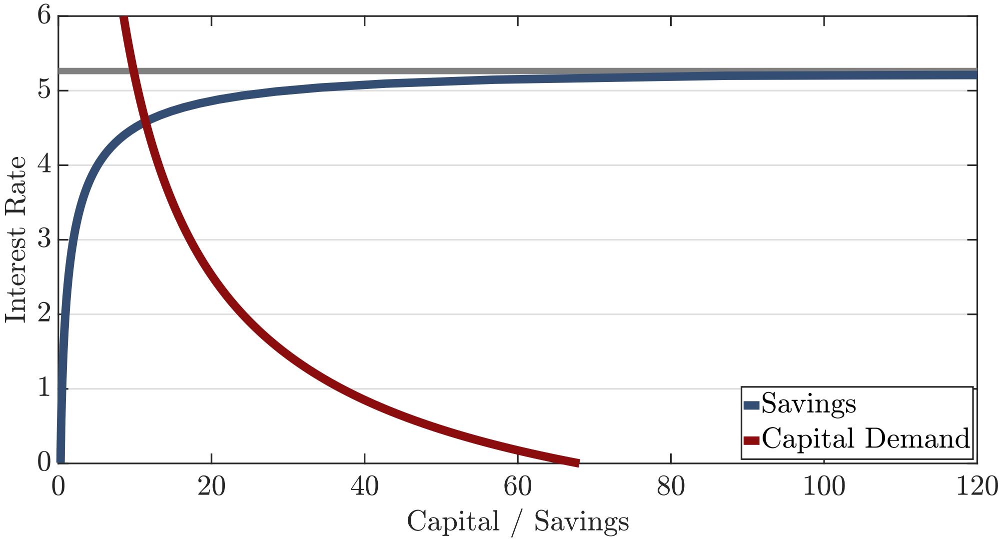
$$r = \alpha\,(K/L)^{\alpha-1} - \delta, \qquad w = (1-\alpha)\,(K/L)^{\alpha}$$

The Aiyagari "Savings–Interest" Picture

Think of two curves in (r, K) space.

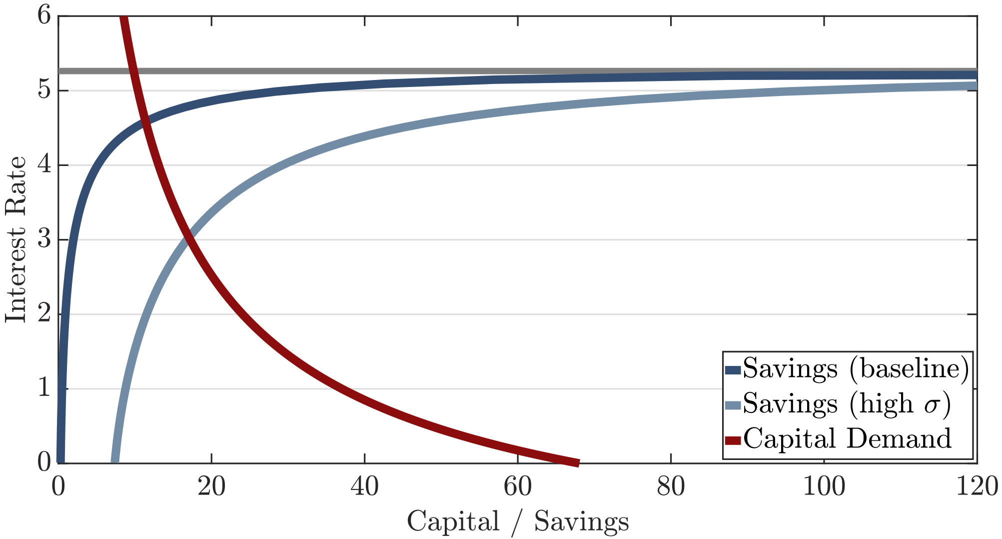
- **Household asset supply:** $K^s(r)$ from $\big(r, w(r)\big) \rightsquigarrow a'_r \rightsquigarrow \mu_r \rightsquigarrow \int a \, d\mu_r$
- **Firm capital demand:** $K^d(r)$ from $r + \delta = \alpha(K/L)^{\alpha-1}$
- **Equilibrium:** $K^s(r) = K^d(r)$

Equilibrium Graph



Comparative Statics

Comparative Statics I: Increase Income Risk σ

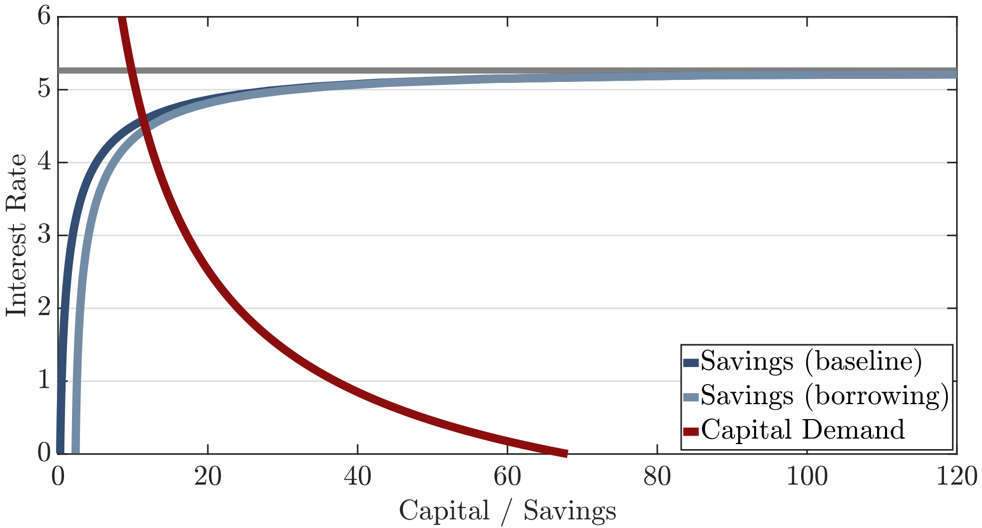


Comparative Statics I: Increase Income Risk σ

Higher idiosyncratic innovation variance increases precautionary motives.

- More risk $\rightsquigarrow$ higher desired buffer stock
- $K^s(r)$ shifts right: more saving at each r
- New equilibrium: $K\uparrow$, $r\downarrow$, $w\uparrow$

Comparative Statics II: Loosen Borrowing Limit $\underline{a}$

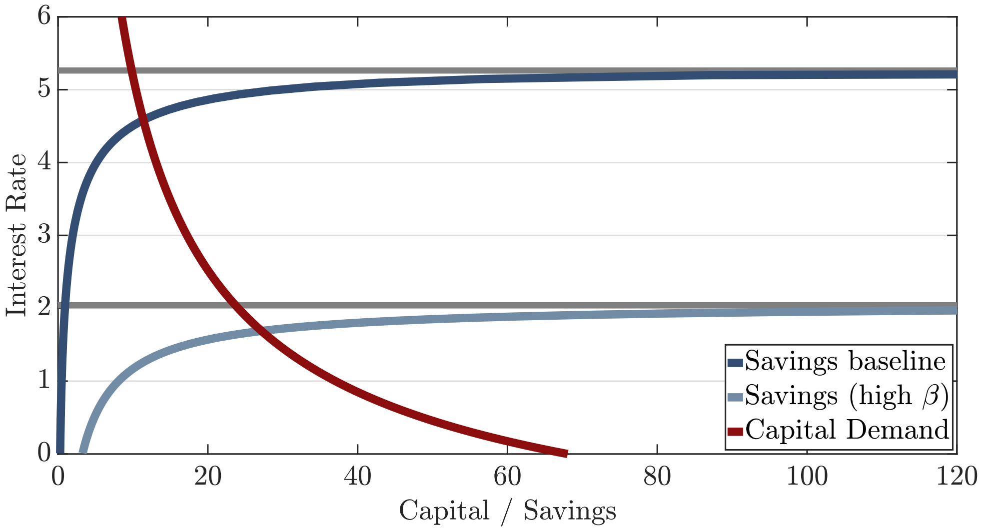


Comparative Statics II: Loosen Borrowing Limit $\underline{a}$

A looser constraint makes bad states less painful.

- Less households hit (or approach) the constraint
- Households are less concerned about borrowing $\rightsquigarrow$ precautionary saving weakens
- But, looser $\underline{a}$ also makes smoothing at low wealth more feasible

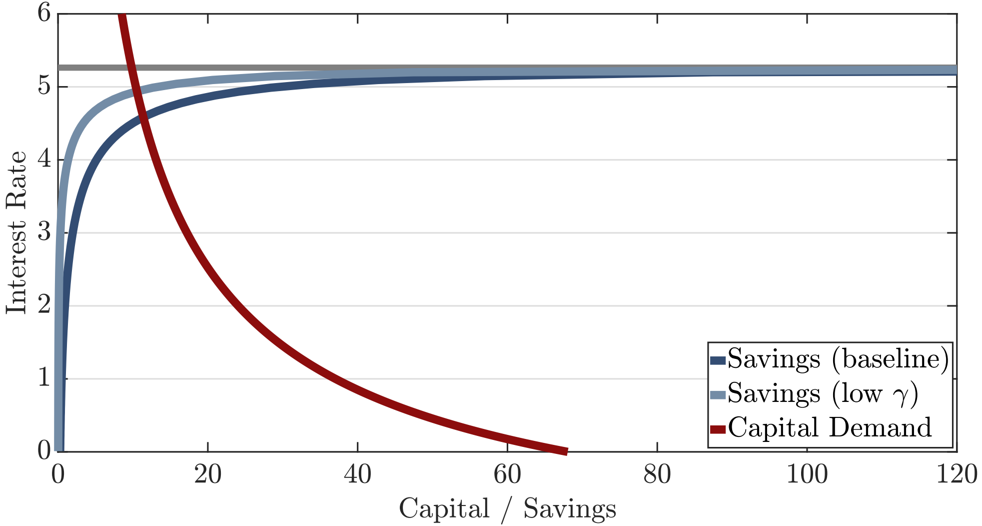
Comparative Statics III: Patience β



Comparative Statics III: Patience β

- Higher β raises willingness to postpone consumption
- $K^s(r)$ shifts right and down (there is a lower ceiling)
- Equilibrium: higher K, lower r, higher w

Comparative Statics IV: Risk Aversion γ

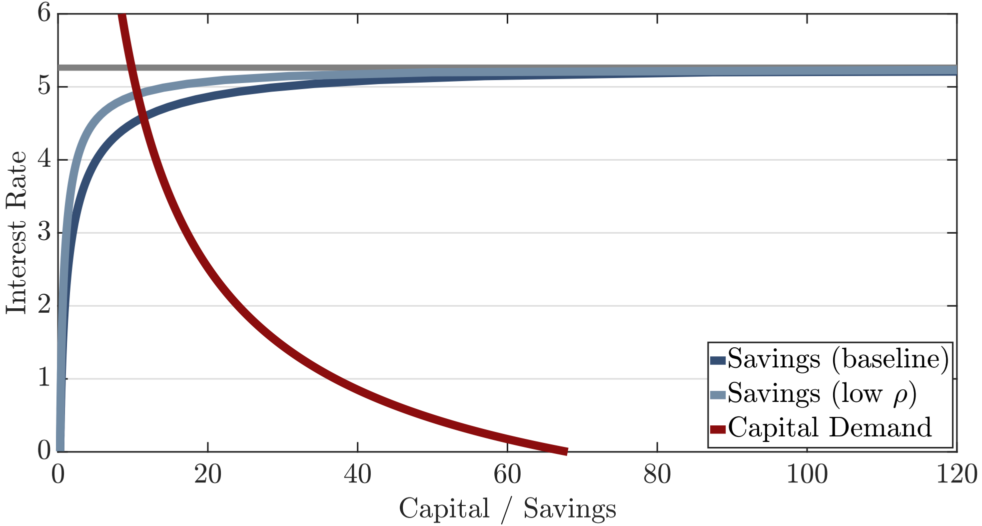


Comparative Statics IV: Risk Aversion γ

Risk aversion changes both substitution and precautionary forces.

- Lower γ decreases prudence $\rightsquigarrow$ weaker precautionary motive
- Lower γ increases intertemporal substitution $\rightsquigarrow$ savings respond more to interest
- Net effect of lowering γ: more often than not $K^s(r)$ shifts left

Comparative Statics V: Persistence ρ_z



Comparative Statics V: Persistence ρ_z

Persistence changes how long bad shocks last.

- Lower ρ_z makes low z less persistent
- "Bad times do not last as long" $\rightsquigarrow$ less self-insurance
- When lowering ρ_z, often, precautionary saving weakens and $K^s(r)$ shifts left